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THE PROPERTIES OF SOLUTIONS OF A CLASS OF ISOPERIMETRIC PROBLEMS OF STABILITY OPTIMIZATION†

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A class of isoperimetric problems of stability optimization is considered. These arise, for example, when maximizing the Euler force in the destabilization of a column (rod) of varying cross-section and given volume (Lagrange's problem). It is well known that an extremum which depends on the form of the boundary conditions can be achieved for both simple and double eigenvalues. A class of problems is identified for which a global maximum is found at a simple eigenvalue. The possibility of achieving a local extremum for the first (simple) eigenvalue at stationary points is analysed qualitatively in terms of the parameter values and the form of the boundary conditions.

1. STATEMENT OF THE PROBLEM

The problem of finding the thickness distribution for a rod of given volume with a maximum destabilization force reduces [1] to investigating the extremal properties of the first (least) eigenvalue of a boundary-value problem for a second-order eigenvalue. Boundary-value problems of this type also arise when studying other physical and mathematical problems [2–6]. A complete investigation of this problem is therefore relevant for general self-adjoint boundary conditions.

Consider the following self-adjoint boundary-value problem for eigenvalues

$$y''(x)+\lambda h^{-p}(x)y(x)=0, \quad 0<x<1 \tag{1.1}$$

$$\alpha_1 y(0)+\alpha_2 y'(1)+\alpha_3 y(1)=0, \quad \alpha_1 y(1)-\alpha_2 y'(0)+\alpha_4 y(0)=0, \quad \alpha_i=\text{const}, \quad i=1,2,3,4 \tag{1.2}$$

Here p is a real number ($p \neq 0$), and $h(x)$, ($x \in [0, 1]$) is a function which satisfies the following conditions

$$h(x)\in L_\infty, \quad 0\leq h(x)<K, \quad \langle h(x)\rangle=1 \tag{1.3}$$

Here and below angle brackets denote integration with respect to x from 0 to 1, K is a fairy large positive number, the order of degeneracy when the function $h(x)$ vanishes within the interval [0, 1] does not exceed $1/(p + 2)$, i.e. if $h(x_0) = 0$ $(0 < x_0 < 1)$, then $h(x) = O((x-x_0)^\gamma)$ where $\gamma < 1/(p + 2)$ and at the ends of the interval degeneracies with orders not exceeding unity are allowed. The set of all functions satisfying the above conditions will be denoted by Q_p.

In the rod stability problem the function h is the area of cross-section, and the parameter p is usually taken to be 1, 2 or 3. Boundary conditions (1.2) can take one of four forms

$$\begin{aligned}&(1)\ y(0)=y(1)=0;\ (2)\ y'(0)+y(0)=y(1)=0;\ (3)\ y'(0)=y(1)=0;\\&(4)\ y'(0)=y'(1), \quad y(1)=y(0)+y'(0)\end{aligned} \tag{1.4}$$

These are the cases when both ends of the rod are freely supported (1), when one is clamped and the other is freely supported (2), when one is clamped and the other is free (3), and when both ends are clamped (4). The first (non-zero) eigenvalue of problem (1.1) with one of the boundary conditions

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(1.4) can be interpreted as the force at which the rod loses stability. From a broader point of view it is interesting to extend the Lagrange problem to the case of general self-conjugate boundary conditions of the form (1.2) for any values of the parameter $p \neq 0$.

All this enables us to formulate the following extremal problem: it is required to choose a function $h(x) \in Q_p$ such that the least (non-zero) eigenvalue attains its maximum (minimum).

Naturally, the problem first arises of the existence of such functions and of methods of finding them. Solutions of this problem have been obtained [1, 7–10] for $p = 1, 2, 3$ and boundary conditions (1.4). The problem of accurate estimates of the least eigenvalue λ_1 with boundary conditions $y(0) = y(1) = 0$ has been studied for a range of values of the parameter p [4, 5].

The aim of this paper is to obtain a qualitative answer to the following question: for a given value of the parameter p can a local maximum (minimum) of the first (least) eigenvalue $\lambda_1(h)$ be reached if that eigenvalue is simple?

2. A SPECTRAL ANALYSIS OF THE PROBLEM

We shall seek a weak solution of boundary-value problem (1.1), (1.2) with a given function $h(x) \in Q_p$ To do this we consider the space H of functions satisfying boundary conditions (1.2) with norm

$$\|y\|_H = (\langle (y')^2 \rangle + y^2(0) + y^2(1))^{1/2}$$

The function $y(x) \in H$ is a weak solution of problem (1.1), (1.2) if for any function $z(x) \in H$ we have the equality

$$\langle y'(x) z'(x) \rangle = \lambda \langle y(x) z(x) h^{-p}(x) \rangle$$

For an arbitrary element $h(x) \in Q_p$ we consider the space of functions $L^h_{2,p}$ with norm

$$\|y\|_{h,p} = \langle h^{-p}(x) y^2(x) \rangle^{1/2}$$

From the assumptions and conditions (1.3) we have $L^h_{2,p} \subset L^h_{2,p+1} \subset L^h_{2,p+2}$. On the other hand, the set H is included in $L^h_{2,p+2}$. Indeed

$$y(x) = \int_0^x y' dx + y(0)$$

Hence

$$\frac{y^2(x)}{2} \leqslant \left(\int_0^x y' dx \right)^2 + y^2(0) \leqslant \langle (y')^2 \rangle + y^2(0) + y^2(1)$$

We multiply the latter inequality by the functions $h^{-(p+2)}(x)$ and integrate the result from 0 to 1, thereby obtaining

$$\|y\|^2_{h,\,p+2} \leqslant r^2 \|y\|^2_H \quad (r^2 = 2\langle h^{-(p+2)} \rangle, \quad h \in Q_p)$$

It follows from well-known results that the inclusion of H in $L^h_{2,p+2}$ and $L^h_{2,p}$ is compact [11].

It is important to note that in a number of cases the boundary-value problem (1.1), (1.2) has a zero eigenvalue. This can be simple, and we then have

$$\alpha_1 = -\alpha_3 = -\alpha_4, \quad \alpha_1 = \alpha_2 = -\alpha_3 \tag{2.1}$$

or else double, and then

$$\alpha_1 = \alpha_2 = -\alpha_3 = -\alpha_4 \tag{2.2}$$

In particular, the last type of boundary condition in (1.4) corresponds to the case (2.2).

If $\alpha_1 = 0$, then (1.2) are Sturm-type boundary conditions and the spectrum of the problem consists of simple eigenvalues. For example, the first, second and third conditions in (1.4) are of this type.

However, if $\alpha_1 \neq 0$, then [12, 13] the entire set of eigenvalues can be distributed in the form of two sequences $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$, $(\bar{\lambda}_0 <)\ \bar{\lambda}_1 < \bar{\lambda}_2 < \ldots$ that increase without limit with $\lambda_n \leqslant \bar{\lambda}_n < \lambda_{n+1}$. In this case the appearance of double eigenvalues is therefore possible. The last kind of condition in (1.4) is of this type.

Summarizing all of the above, we can formulate the following assertion.

For any function $h(x) \in Q_p$ a denumerable number of eigenfunctions $\{y_i\}_{i=1}^{\infty}$ from the space H exists that are weak solutions of boundary-value problem (1.1), (1.2). If conditions (2.1) and (2.2) are not satisfied, these functions correspond to a sequence of non-zero eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots$.

The maximum multiplicity of the eigenvalues does not exceed two, and in the case when $\alpha_1 = 0$ the spectrum is simple. The eigenfunctions constitute a complete system both in the space H and the space $L^h_{2,p+i}$ $(i = 0, 1, 2)$ and can be chosen so that the following normalization conditions are satisfied

$$\langle h^{-p} y_i y_j \rangle = \delta_{ij}, \quad i,j = 1,2,\ldots \tag{2.3}$$

where δ_{ij} is the Kronecker delta.

If conditions (2.1) or (2.2) are satisfied, problem (1.1), (1.2) has a simple or a double zero eigenvalue. In the case of (2.2) the eigenfunctions corresponding to this value have the form

$$y_{10}(x) = c_1, \quad y_{20}(x) = c_2 x + c_3 \tag{2.4}$$

If the constants c_1, c_2, c_3 are chosen so that

$$c_1 = \gamma_0^{-1/2}, \quad c_2 = -\gamma_1 \left(\frac{\gamma_0}{\gamma_0\gamma_2 - \gamma_1^2} \right)^{1/2}, \quad c_3 = \left(\frac{\gamma_0}{\gamma_0\gamma_2 - \gamma_1^2} \right)^{1/2} \tag{2.5}$$

$$\left(\gamma_0 = \langle h^{-p} \rangle, \ \gamma_1 = \langle x h^{-p} \rangle, \ \gamma_2 = \langle x^2 h^{-p} \rangle \right)$$

then together with (2.3) the conditions

$$\langle h^{-p} y_{0k} y_i \rangle = 0, \ \langle h^{-p} y_{0k} y_{0s} \rangle = \delta_{ks}, \quad k,s = 1,2, \ i = 1,2,\ldots \tag{2.6}$$

are satisfied.

3. VARIATIONAL ANALYSIS OF THE PROBLEM

Let $h(x) \in Q_p$. We consider a function $\delta h(x)$ such that $h_t(x) = h(x) + t\delta h(x) \in Q_p$. Here $|t| \leqslant t_0$ where t_0 is a sufficiently small number. Using results on spectrum perturbations for self-adjoint operators in a Hilbert space [14] we find that the first eigenvalue and the first eigenfunction of the perturbed boundary-value problem (1.1), (1.2) can be expanded in a power series in the small parameter t

$$y_1^t(x) = y_1(x) + t\upsilon_1(x) + t^2 \upsilon_2(x) + o(t^2)$$
$$\lambda_1^t = \lambda_1 + t\mu_1 + t^2 \mu_2 + o(t^2)$$

Here μ_1 μ_2 are some numbers and $\upsilon_1(x)$, $\upsilon_2(x)$ are functions in $L^h{}_{2,p+2}$ which are to be determined.

We substitute the expansions obtained into Eq. (1.1) and equate coefficients of equal powers of t. We then obtain the following sequence of boundary-value problems

$$y_1''(x) + \lambda_1 h^{-p}(x) y_1(x) = 0 \tag{3.1}$$

$$\upsilon_1''(x)+\lambda_1 h^{-p}(x)\upsilon_1(x)=\lambda_1 ph^{-(p+1)}(x)y_1(x)\delta h(x)-\mu_1 h^{-p}(x)y_1(x) \tag{3.2}$$

$$\upsilon_2''(x)+\lambda_1 h^{-p}(x)\upsilon_2(x)=-\tfrac{1}{2}\lambda_1 p(p+1)h^{-(p+2)}(x)y_1(x)\delta h^2(x)+$$
$$+\lambda_1 ph^{-(p+1)}(x)\upsilon_1(x)\delta h(x)+\mu_1 ph^{-(p+1)}(x)y_1(x)\delta h(x)-\mu_1\upsilon_1(x)h^{-p}(x)-\mu_2 h^{-p}(x)y_1(x) \tag{3.3}$$

where the functions $y_1(x)$, $\upsilon_1(x)$, $\upsilon_2(x)$ satisfy the boundary conditions (1.2). We also note that the derivation of (3.1)–(3.3) used the representation

$$h_t^{-p}(x)=h^{-p}(x)-tph^{-(p+1)}(x)\delta h(x)+\tfrac{1}{2}p(p+1)h^{-(p+1)}(x)\delta h^2(x)+o(t^2)$$

Scalar-multiplying Eq. (3.2) by the function $y_1(x)$ in L_2 and using the self-adjointness of boundary-value problem (1.1), (1.2) an the normalization conditions (2.3), we obtain

$$\mu_1(h,\delta h)=\lambda_1 p\langle h^{-(p+1)}y_1^2\delta h\rangle \tag{3.4}$$

Formula (3.4) is the first directional (Kato) derivative of the functional $\lambda_1(h)$. In order to find a representation of the second Kato derivative we expand the function $\upsilon_1(x)$ in terms of a system of eigenfunctions of problem (1.1), (1.2).

If problem (1.1), (1.2) has no zero eigenvalue, then

$$\upsilon_1(x)=\sum_{i=1}^{\infty} a_i y_i(x)$$

We substitute this expansion into Eq. (3.2) and then multiply it sequentially in L_2 by the eigenfunctions $y_2(x), y_3(x)\ldots$. Using conditions (2.3), we find that

$$a_i=-\lambda_1(\lambda_i-\lambda_1)^{-1}p\langle h^{-(p+1)}y_i y_1\delta h\rangle,\quad i=2,3,\ldots$$

The coefficient a_1 can be determined from condition (2.3) when $i=j=1$.

Indeed, substituting the expansions in powers of t obtained above into (2.3) with $i=j=1$, we obtain

$$2\langle h^{-p}\upsilon_1 y_1\rangle-p\langle h^{-(p+1)}y_1^2\delta h\rangle=0 \tag{3.5}$$

From this it follows that

$$a_1=p\langle h^{-(p+1)}y_1^2\delta h\rangle/2 \tag{3.6}$$

If the function $\upsilon_1(x)$ is substituted into Eq. (3.3) and then scalar-multiplied in L_2 by the function y_1, by using the same arguments as when finding μ_1 together with formulae (3.5) and (3.6) we obtain

$$\mu_2(h,\delta h)=-\lambda_1 p(p+1)\langle h^{-(p+2)}y_1^2\delta h^2\rangle/2+\lambda_1 p^2\langle h^{-(p+1)}y_1^2\delta h\rangle^2-$$
$$-\lambda_1^2 p^2\sum_{i=2}^{\infty}(\lambda_i-\lambda_1)^{-1}\langle h^{-(p+1)}y_i y_1\delta h\rangle^2 \tag{3.7}$$

Note that because $y_i(x)\in L^h_{2,p+2}$, $(i=1,2,\ldots)$, all the integrals in (3.4) and (3.7) are well-defined.

We will now consider the case when boundary-value problem (1.1), (1.2) has a zero eigenvalue. Suppose, for example, that condition (2.2) is satisfied, i.e. $\lambda_0=0$ is a double eigenvalue. Then

$$\upsilon_1(x)=a_{01}y_{01}(x)+a_{02}y_{02}(x)+a_1 y_1(x)+a_2 y_2(x)+\ldots$$

Repeating all the preceding arguments, we obtain in this case an equation which differs from (3.7) in that terms of the form

$$\lambda_1 p^2\sum_{k=1}^{2}\langle h^{-(p+1)}y_{0k}y_1\delta h\rangle^2 \tag{3.8}$$

are added to the right-hand side.

However, if problem (1.1), (1.2) only has a simple zero eigenvalue, then one of the functions $y_{01}(x)$ or $y_{02}(x)$ must be omitted from the right-hand side of formula (3.7) when (3.8) is included. Here the number of terms with a plus sign is reduced by one.

It is important to note that formulae (3.4), (3.7) and (3.8) give the first and second Kato derivatives of the functional $\lambda_1(h)$ only when $\lambda_1(h)$ is a simple eigenvalue.

We will now formulate the necessary conditions for the extremum of the functional $\lambda_1(h)$. Using standard methods of the variational calculus, it can be shown that they are of the following form [1]

$$h^{-(p+1)}(x)y_1^2(x)=1,\quad x\in(0,1) \tag{3.9}$$

Suppose that (3.9) is satisfied. Then sign-definiteness of the forms (3.7) and (3.8) with respect to functions $\delta h(x)$ satisfying the conditions

$$\langle\delta h\rangle=0,\quad \delta h(x)\in L_\infty \tag{3.10}$$

is a necessary condition for a second-order extremum of the functional $\lambda_1(h)$. However, if for standard $h(x)$ and $y_1(x)$ variations $\delta h(x)$ exist satisfying (3.10) such that $\mu_2(h, \delta h) > 0$ (< 0), then one can assert that the functional $\lambda_1(h)$ does not reach a maximum (minimum) at its stationary points. All this makes the use of formulae (3.7) and (3.8) suitable for proving assertions about local extrema not being reached at stationary points. There are exceptional cases when $\mu_2(h, \delta h) > 0$ (<0) for all $h(x) \in Q_p$ and all functions $\delta h(x) \in L_\infty$. In this case one can assert that the functional $\lambda_1(h)$ [15] is strictly convex (concave) an that consequently any local extremum will be global on the entire set.

4. THE CASE $0 < p \leqslant 1$

The following result holds irrespective of the type of boundary condition (1.2).

Theorem 4.1. The functional $\lambda_1(h)$ (where λ_1 is a simple eigenvalue) is strongly concave when $0 < p < 1$ and concave when $p = 1$.

Proof. We introduce the auxiliary function

$$\psi(x)=y_1(x)h^{-1}(x)\delta h(x),\quad \delta h(x)\in L_\infty$$

We shall prove that $\psi(x) \in L^h_{2,p}$. Indeed

$$\langle\psi^2h^{-p}\rangle=\langle y_1^2h^{-(p+2)}\delta h^2\rangle\leqslant\Big(\max_{x\in[0,1]}\delta h^2\Big)\|y_1\|^2_{h,\,p+2}<\infty$$

We expand the function $\psi(x)$ in terms of the eigenfunction system $y_{01}(x), y_{02}(x), y_1(x), y_2(x), \ldots$ of problem (1.1), (1.2) which is complete in $L^h_{2,p}$. We recall that the functions $y_{01}(x)$ and $y_{02}(x)$ defined in (2.4)–(2.6) occur in the expansion only in the case when the boundary-value problem (1.1), (1.2) admits of the existence of a zero eigenvalue. We have

$$\psi(x)=\sum_{k=1}^{2}d_{0k}y_{0k}(x)+\sum_{i=1}^{\infty}d_iy_i(x)$$

The Fourier coefficients d_{0k} and d_i are given by the formulae

$$d_{0k}=\langle h^{-p}\psi y_{0k}\rangle,\quad d_i=\langle h^{-p}\psi y_i\rangle,\quad k=1,2,\quad i=1,2,\ldots$$

Consequently

$$\|\psi\|^2_{h,p}=\sum_{k=1}^{2}d_{0k}^2+\sum_{i=1}^{\infty}d_i^2=\langle h^{-(p+2)}y_1^2\delta h^2\rangle$$

Using the last two equalities and bearing in mind (3.8), we find that formula (3.7) can be written in the following form

$$\mu_2(h,\delta h) = -\lambda_1 p(p+1)\|\psi\|^2_{h,p}/2 + \lambda_1 p^2(d_{01}^2 + d_{02}^2 + d_1^2) - p^2\sum_{i=2}^{\infty}(\lambda_i - \lambda_1)^{-1} d_i^2$$

Because

$$\lambda_i > \lambda_1, \quad i = 2,3,\ldots, \quad d_1^2 + d_{01}^2 + d_{02}^2 \leqslant \|\psi\|^2_{h,p}$$

we have the inequality

$$\mu_2(h,\delta h) \leqslant \lambda_1 p(p-1)\|\psi\|^2_{h,p}/2$$

The last expression is strictly negative when $0 < P < 1$ for any $\delta h \in L_\infty$.
When $p = 1$ we find that $\mu_2(h, \delta h) \leqslant 0$ which it was required to prove.

5. THE CASES $p < 0$ AND $p > 1$. BOUNDARY-VALUE PROBLEM WITHOUT ZERO EIGENVALUES

We shall assume boundary-value problem (1.1), (1.2) has no zero eigenvalues, i.e. conditions (2.1), (2.2) are not satisfied.

Theorem 5.1. When $p < -1$ and $p > 1$ the functional $\lambda_1(h)$ does not reach a local minimum, while when $-1 < p < 0$ there is a local maximum at the stationary points if $\lambda_1(h)$ is a simple eigenvalue.

Proof. We consider the set of admissible variations $\delta h \in L_\infty$ satisfying conditions (3.10). Since the necessary condition (3.9) is satisfied

$$\langle h^{-(p+1)} y_1^2 \delta h\rangle = \langle \delta h\rangle = 0 \tag{5.1}$$

Taking this into account it follows from (3.7) that $\mu_2(h, \delta h) < 0$, $\delta h \neq 0$ when $p < -1$ or $p > 0$ because the last sum in (3.7) is obviously non-negative.

We will now consider the case $-1 < p < 0$. We introduce the following system of functions

$$g_1(x) = 1, \quad g_i(x) = h^{-(p+1)}(x) y_1(x) y_i(x), \quad i = 2,3,\ldots$$

Here $\{y_i(x)\}_{i=1}^{\infty}$ is a system of eigenfunctions of problem (1.1), (1.2) with a stationary $h(x)$ satisfying condition (3.9). The functions $g_i(x)$ belong to the space L_1. Indeed

$$\langle h^{-(p+1)}|y_1\|y_i|\rangle \leqslant \langle h^{-(p+1)} y_1^2\rangle^{1/2} \langle y_i^2 h^{-(p+1)}\rangle^{1/2} \leqslant \|y_i\|^2_{h,p+2} < \infty$$

Here we have used condition (3.9) from which $\langle h^{-(p+1)} y_1^2\rangle$.
The system $\{g_i(x)\}_{i=1}^{\infty}$ is linearly independent.

Suppose the contrary. Then constants $\delta_1, \delta_2 \ldots$ exist such that

$$\sum_{i=2}^{\infty} \delta_i g_i(x) + \delta_1 = 0$$

We multiply this equality by the function $h(x)$ and integrate from 0 to 1. By conditions (2.3) and (1.3) we obtain $\delta_1\langle h(x)\rangle = \delta_1 = 0$. But we then have the equality

$$y_1(x)\sum_{i=2}^{\infty} \delta_i y_i(x) = 0$$

which is impossible by the linear independence of the eigenfunctions $y_i(x)$, $i = 2, 3 \ldots$.

We consider the linear space Z_N generated by the functions

$$g_1 = 1, \quad g_i = y_1 y_i h^{-(p+1)}, \quad i = 1,2,\ldots, N-1, N+2,\ldots$$

$$g_N = h^{-(p+1)} y_N y_1 \notin Z_N$$

We use the separability theorem. An element $\delta h^0 \in L_\infty$ exists such that the linear functional

$$F(g) = \langle g\delta h^0 \rangle, \quad g \in L_1$$

is equal to zero if $g \in Z_N$ and $F(g_N) \neq 0$ when $g_N \notin Z_N$.

We note that because $g_1 = 1$ lies in the set Z_N, condition (5.1) is satisfied. Expression (3.7) for the second Kato derivative will not contain a second right-hand term for the element δh^0 obtained, and from the entire infinite sum that constitutes the third term only the term corresponding to $i = N$ remains. Because

$$\langle h^{-(p+1)} y_N y_1 \delta h^0 \rangle^2 \leq \langle h^{-p} y_N^2 \rangle \langle h^{-(p+2)} y_1^2 (\delta h^0)^2 \rangle$$

taking into account the normalization condition (2.3), we find from (3.7) that

$$\mu^2(h, \delta h^0) \geq -\lambda_1 p \left(\frac{p+1}{2} + \frac{\lambda_1 p}{\lambda_N - \lambda_1} \right) \langle h^{-(p+2)} y_1^2 (\delta h^0)^2 \rangle \tag{5.2}$$

The last expression will be strictly positive if

$$-1 + 2k_N / (1 + k_N)^{-1} < p < 0 \quad (k_N = \lambda_1 / \lambda_N)$$

Since $\lambda_N \to \infty$ when $N \to \infty$, by choosing sufficiently large N one can arrive at a right-hand side for (5.2) that is positive when $-1 < p < 0$. Hence the functional does not reach a local maximum for the given values of p.

If the coefficient $\alpha_1 = 0$ in (1.2), the boundary-value problem (1.1), (1.2) is of Sturm type and its spectrum is simple. Hence the requirement for simple eigenvalues in the conditions of Theorem 5.1 can be omitted.

6. THE CASES $p < 0$ AND $p > 1$. BOUNDARY-VALUE PROBLEM WITH ZERO EIGENVALUE

The following result holds.

Theorem 6.1. Suppose that boundary-value problem (1.1), (1.2) has a zero eigenvalue, that condition (3.9) is satisfied for the function $h(x)$ and that when $p > 1$ the inequality

$$h(x) \geq \delta^2 > 0, \quad x \in (0,1) \tag{6.1}$$

holds.

Then when $p < 0$ and $p > 1$ the functional $\lambda_1(h)$ does not reach a local extremum at the stationary points if $\lambda_1(h)$ is a simple eigenvalue.

Proof. We will first consider the case when the first condition in (2.1) is satisfied, case (2.2) being considered similarly. We put

$$\delta h = h^{-p}(x) y_1(x) \tag{6.2}$$

We will show that the variation δh is admissible. Because the eigenfunction $y_{01}(x) = c_1$ (the constant c_1 being defined by formulae (2.5)) is orthogonal to the first eigenfunction in $L^h_{2,p}$, we have $\langle \delta h \rangle = \langle h^{-p} y_1 \rangle = 0$. Thus condition (3.10) is satisfied. We will show that $\delta h(x)$ is a bounded function. Indeed, from the necessary condition for an extremum (3.9) we have

$$\delta h^2(x) = h^{-2p}(x) y_1^2(x) = h^{1-p}(x)$$

The last function is bounded for all $p < 0$, and also for all $p > 1$ if condition (6.1) is satisfied.

We substitute δh in the form (6.2) into expression (3.7) for the second Kato derivative of the functional $\lambda_1(h)$ and bear (3.8) in mind. We again use condition (3.9). Then for $i = 2, 3, \dots$ we have the equality

$$\langle h^{-(p+1)} y_i y_1 \delta h\rangle = \langle h^{-p} y_i\rangle = 0 \tag{6.3}$$

because the eigenfunctions y_i are orthogonal to the eigenfunction $y_{01}(x) = c_1$.

On the other hand

$$\langle h^{-(p+1)} y_{01} y_1 \delta h\rangle = \langle h^{-p}\rangle^{1/2} \tag{6.4}$$

Here we have used formulae (2.5) for the value of the constant c_1 and the normalization condition (2.6).

From (6.3) and (6.4) we find that formula (3.7), by virtue of (3.8), can be represented in the form

$$\mu_2(h,\delta h) = \lambda_1 p(p-1)\langle h^{-p}\rangle/2 \tag{6.5}$$

Here we have used the equality

$$\langle h^{-(p+2)} y_1^2 \delta h^2\rangle = \langle h^{-2(p+1)} y_1^4 h^{-p}\rangle = \langle h^{-p}\rangle$$

which follows from (6.2) and condition (3.9).

From (6.5) we find that $\mu_2(h, \delta h) > 0$ when $p < 0$ or $p > 1$, and that consequently the functional $\lambda_1(h)$ does not reach a local maximum at stationary points.

These values of the parameter p do not admit of a local minimum either.

Hence, as in the case of Theorem 5.1, we consider the system of functions

$$g_1 = 1, \quad g_i = y_1 y_i h^{-(p+1)}, \quad i = 1,2,3\dots$$

adding to it the functions $g_0 = y_{10} y_1 h^{-(p+1)}$. As before, it can be proved that the functions $g_i \in L_1$, and the system $\{g_i\}_{i=0}^{\infty}$ is linearly independent. Using the separability theorem we construct a functional

$$F(g) = \langle g \delta h^0\rangle, \quad g \in L_1, \quad \delta h^0 \in L_\infty$$

such that

$$F(g_0) = 0, \quad F(g_1) = 0, \quad F(g_i) \neq 0, \quad i = 2,3,\dots$$

We note that condition (5.1) is satisfied here because condition (3.9) holds.

The second Kato derivative (3.7) at the element has the form

$$\mu_2(h,\delta h^0) = -\lambda_1 p(p+1)\langle h^{-(p+2)} y_1^2 (\delta h^0)^2\rangle - p^2 \lambda_1 \sum_{i=2}^{\infty} (\lambda_i - \lambda_1)^{-1} \langle h^{-(p+1)} y_1 \delta h^0\rangle^2$$

Hence $\mu_2(h, \delta h^0) < 0$ when $p \leqslant -1$ or $p > 0$. From the result of Theorem 4.1 we know that when $0 < p \leqslant 1$ the functional $\lambda_1(h)$ is concave. From this we conclude that the functional $\lambda_1(h)$ does not reach a local minimum when $p < 0$ or $p > 1$ when $\lambda_1(h)$ is a simple eigenvalue.

Remark. If $\alpha_1 \neq 0$ and boundary-value problem (1.1), (1.2) has a zero eigenvalue, it follows from Theorem 6.1 that when $p < 0$ or $p > 1$ a local maximum or minimum of the functional $\lambda_1(h)$ can only be reached at a double λ_1. If however $\alpha_1 = 0$, a local extremum is in general not reached at stationary points because in this case the spectrum is simple.

7. CONCLUSION

The results of this paper show that the multiplicity property of an extremal eigenvalue depends strongly on the possibility of the appearance of zero eigenvalues in boundary-value problem (1.1), (1.2). It is this feature that distinguishes the case of rigid attachment in the Lagrange problem from other boundary conditions.

Note that the initial solution of this problem [1] had points of degeneracy (zeros) of the function $h(x)$, with the maximum being reached at a simple λ_1. Indeed, as has been shown [7], the maximum is reached at a double λ_1. The necessary conditions for an extremum in this case were obtained [16] and extended to the general case [17]. On the basis of these conditions the problem was solved [8] for the cases $p = 1, 2, 3$. When $p = 2, 3$ the maximum is achieved at a double λ_1, and the optimal distribution $h(x)$ does not vanish, which agrees with the assertion of Theorem 6.1. Theorem 4.1 shows that in the $p = 1$ case the maximum is reached at a simple eigenvalue. Note that the results obtained in [8] were repeated in [9, 10], and in [10] the case of boundary conditions (1.4) with $p > 0$ was also studied.

Results in [4, 5] which give exact estimates for the least eigenvalue λ_1 are in good agreement with the conclusion of Theorem 5.1. Nearly sufficient extremum conditions were also considered in [18] for the case $p = 2$ and boundary conditions (1.4).

It should be noted that all the results obtained above can be significantly strengthened if the derivatives (3.7) and (3.8) are strong functional derivatives in the Fréchet sense. In particular, it has been shown [19] that this is the case if, in addition to conditions (1.3), the function $h(x)$ has a bounded square-integrable derivative. Here the problem of the extremal properties of the first eigenvalue of system (1.1), (1.2) acquires an essentially new form requiring special consideration.

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